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SOME CONSEQUENCES OF PISIER'S APPROACH TO INTERPOLATION

BY

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ABSTRACT

A recent result of Pisier on K-functionals is exploited in the context of function spaces defined from singular integrals. The main consequences appear for H^p spaces on the complex ball $(1 \leq p \leq \infty)$ and spaces of differentiable functions in several variables. It turns out that the complex variable arguments used in [P1] may in certain cases be substituted for real variable methods, which of course have a wider range of applicability. We discuss interpolation of vector valued Hardy spaces on the ball and prove that $L^1(S)/H^1(B)$ satisfies Grothendieck's theorem, similarly as in the disc case.

1. Introduction

The aim of this note is to complement some of Pisier's results obtained in the recent papers [P1], [P2]. The paper [P1] deals with interpolation of the classical Hardy spaces H^p $(1 \leq p \leq \infty)$ on the disc D and non-commutative generalizations. In [P2], a new proof is given of several results from [B1] and in particular the fact that the quotient space $L^1(\mathbb{T})/H^1(D)$ satisfies Grothendieck's theorem and has cotype 2 (the reader should consult [B3] and/or expository work such as [P3] for definitions and discussions of these concepts). Both papers [P1], [P2] make crucial use of a formal algebraic lemma which in some sense permits one to dualize the problem of studying K-functionals. Let us recall briefly these concepts and some results from [P1].

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Let A_0, A_1 be a couple of interpolation (Banach) spaces. For $x \in A_0 + A_1$ and $t > 0$, let

$$(1.1)\qquad K_t(x, A_0, A_1) = \inf(\|x_0\|_{A_0} + t\|x_1\|_{A_1} | x = x_0 + x_1, x_0 \in A_0, x_1 \in A_1).$$

For $x \in A_0 \cap A_1$ and $t > 0$, let

$$(1.2)\qquad J_t(x, A_0, A_1) = \max(\|x_0\|_{A_0}, t\|x_1\|_{A_1}).$$

The (real interpolation) space $(A_0, A_1)_{\theta,p}$ is defined on the space of all x in $A_0 + A_1$ such that $\|x\|_{\theta,p} < \infty$ where

$$(1.3)\qquad \|x\|_{\theta,p} = \left(\int \left(t^{-\theta} K_t(x, A_0, A_1)\right)^p \frac{dt}{t}\right)^{1/p}.$$

In particular, one has

$$(1.4)\qquad L_{p,q} = (L_{p_0}, L_{p_1})_{\theta,q}$$

where $1 \le p_0, p_1, q \le \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$.

Assume next S is a closed subspace of $A_0 + A_1$ and let

$$(1.5)\qquad S_0 = S \cap A_0, \qquad S_1 = S \cap A_1.$$

In [P1], the couple (S_0, S_1) is called K-closed relative to (A_0, A_1) if there is a constant c such that

$$(1.6)\qquad \forall t > 0, \quad \forall x \in S_0 + S_1: \qquad K_t(x, S_0, S_1) \le cK_t(s, A_0, A_1).$$

The lemma we were referring to above is the following fact:

LEMMA 1.7 (see [P1]): *Assume $A_0 \cap A_1$ dense in A_0 and in A_1 and assume there is a subspace $s \subset A_0 \cap A_1$ which is dense in S_0 with respect to A_0 and in S_1 with respect to A_1. Then (S_0, S_1) is K-closed in (A_0, A_1) iff $(S_0^\perp, S_1^\perp)$ is K-closed in (A_0^*, A_1^*).*

Let us sketch the argument for completeness sake.

If $(S_0^\perp, S_1^\perp)$ is K-closed in (A_0^*, A_1^*), then there is an inequality

$$(1.8)\qquad K_t(x, S_0^\perp, S_1^\perp) \le cK_t(x, A_0^*, A_1^*) \qquad \text{for } x \in S_0^\perp + S_1^\perp$$

which by duality implies that there is the following simultaneous lifting property:

$$\begin{aligned}&\forall t>0,\quad \forall x\in Q_0\cap Q_1,\quad \exists \hat{x}\in A_0\cap A_1\\ &\text{such that}\ \ J_t(\hat{x},A_0,A_1)\le cJ_t(x,Q_0,Q_1)\end{aligned}\tag{1.9}$$

denoting $Q_0 = A_0/S_0, Q_1 = A_1/S_1$ the quotient spaces. We have to show (1.6), where c refers to some constant. Take $x = s_0 + s_1 \in S_0 + S_1$ with a decomposition $x = x_0 + x_1,\ x_0 \in A_0,\ x_1 \in A_1$ and

$$\|x_0\|_{A_0} + t\|x_1\|_{A_1} \le 1.\tag{1.10}$$

Write $x_0 - s_0 = s_1 - x_1 = y \in A_0 \cap A_1$ for which, obviously from (1.10),

$$\|\tilde{y}\|_{Q_0} + t\|\tilde{y}\|_{Q_1} \le 1\tag{1.11}$$

using the notation $\tilde{y}$ for the quotient element. Hence $J_t(\tilde{y}, Q_0, Q_1) \le 1$ and property (1.9) yields a vector $z \in A_0 \cap A_1$ satisfying

$$y - z \in S_0 \cap S_1; \qquad J_t(z, A_0, A_1) \le c.\tag{1.12}$$

It follows from the preceding that

$$x = (x_0 - z) + (x_1 + z) = (s_0 + y - z) + (s_1 - y + z) \equiv s_0' + s_1'\tag{1.13}$$

which is an $S_0 + S_1$ decomposition of x where, by (1.10) and (1.12),

$$\|s_0'\|_{A_0} + t\|s_1'\|_{A_1} \le (\|x_0\|_{A_0} + t\|x_1\|_{A_1}) + \|z\|_{A_0} + t\|z\|_{A_1} \le 1 + 2c.\tag{1.14}$$

Consequently

$$K_t(x, S_0, S_1) \le 1 + 2c\tag{1.15}$$

completing the argument.

In [P1], another proof is given of P. Jones's interpolation theorem [J] based on the previous device. The more delicate (and interesting) aspect of that result is when $H^\infty = H^\infty(D)$ appears as an endpoint, thus

$$H^p = (H^{p_0}, H^\infty)_{\theta,p}, \quad 1 \le p_0 < p < \infty, \quad 1/p = (1-\theta)/p_0\tag{1.16}$$

(in the real case, which is the only one we will consider here).

With the notations from (1.7),

$$A_0 = L^{p_0}(\mathbb{T}), \quad A_1 = L^\infty(\mathbb{T}), \quad S_0 = H^{p_0}(D), \quad S_1 = H^\infty(D).$$

The K-functional problem is then reduced to the corresponding problem for the couple $(S_0^\perp, S_1^\perp)$ which in this case identifies with $(H^{p_0'}(D), H^1(D))$, $p_0' = p_0/(p_0-1)$. Proving that $(H^q(D), H^1(D))$ is K-closed in $(L^q(\mathbb{T}), L^1(\mathbb{T}))$ is achieved in [P1] by invoking complex variable techniques, such as factorization. Our aim here is to substitute that part of the proof by real variable methods, leading to K-functional inequalities and interpolation results for the L^p-closures of function spaces which L^2-orthogonal projection is given by a Calderon–Zygmund type kernel on a homogeneous domain. The most interesting case will be that of the H^p-spaces on the ball:

THEOREM 1: *Let $1 \le p_0 < p < p_1 \le \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then denoting $H^p(B)$ the Hardy space on the complex unit ball B of $\mathbb{C}^d$, one has*

$$H^p(B) = (H^{p_0}(B), H^{p_1}(B))_{\theta,p} \tag{1.17}$$

and for $1 \le q \le \infty$

$$H^{p,q}(B) = (H^{p_0}(B), H^{p_1}(B))_{\theta,q}. \tag{1.18}$$

The argument will only use real variable techniques, i.e. the geometry of the boundary, and may be rewritten for strictly pseudo-convex domains.

It should be mentioned that the truly new information consists in the K-functional inequalities. The interpolation phenomena were essentially known to the author and also to S. Kisliakov. Kisliakov worked more specifically on the H^p-spaces $H^p(D^2)$ on the bidisc $D^2 = D \times D$ (cf. [Kis]) which in fact is a more difficult case than the ball. For instance, certain interpolation results involving $p = \infty$ are known in the bidisc case, but at this point no K-functional inequalities in the sense of (1.6) relating to $H^\infty(D^2)$. In the polydisc case D^r, $r \ge 3$, no interpolation results relating to $H^\infty(D^r)$ are known. (See also [Kis] for more details and applications.)

In [P2], a new proof is given of the results on absolutely summing operators on the disc algebra $A(D)$ and $H^\infty(D)$ from [B1]. These results are derived from certain inequalities of the form

$$\pi_q(u) \le c\pi_2(u)^\theta \|u\|^{1-\theta} \tag{1.19}$$

where $u : A \to Y$ is a 2-summing operator on $A = A(D)$ and $2 < q < \infty$. The proof of (1.19) is obtained from interpolation properties of the vector-valued H^∞-spaces $H^\infty_{\ell^p}(D)$ $(2 \le p \le \infty)$ on the disc. Such interpolation results and also (1.19) fail if one replaces D by the ball B_r, $r \ge 2$ or the polydisc D^r $(r \ge 2)$, although the problem of Grothendieck property and cotype for the dual spaces $A(B_r)^*, A(D^r)^*$ is presently still unsettled. The previous considerations will be elaborated later on in the paper. They consist in fact just in recalling earlier work ([B2], [B3], [B4]) on the behavior of linear operators on these algebras.

Using the method of [P2], Xu proved that quotient spaces $L^1(\mathbb{T}^r)/H^1(D^r), r = 2, 3, \dots$ satisfy Grothendieck's theorem, analogously to $L^1(\mathbb{T})/H^1$ ([Xu]). His argument is a nice combination of various techniques in vector-valued harmonic analysis developed over recent years. In the last section of the paper, we will extend his result to the complex ball B_2, thus

THEOREM 2: *$L^2(S)/H^1(B_2)$ has Grothendieck's property (i.e. linear operators acting on this space and ranging in ℓ^2 are 1-summing).*

Observe that for $r \ge 2$ the spaces $L^1(\mathbb{T}^r)/H^1(D^r)$ (resp. $L^1(S)/H^1(B_r)$) do not appear as predual of $H^\infty(D^r)$, resp. $H^\infty(B_r)$.

It is conceivable that the [P2] approach may permit one to progress on the absolutely summing operator problems related to other spaces, such as the space $W^{1,\infty}(\mathbb{T}^2)$ of functions on $\mathbb{T}^2$ with bounded gradient (private communications of S. Kisliakov, 2/91).

2. Proof of Theorem 1 and Generalizations

The argument may in fact be presented with a level of abstraction similar to that of Lemma 1.7.

The main lemma may be formalized for spaces of homogeneous type X, d, μ (X = topological space, d = quasi-distance, $\mu \ge 0$ a finite Borel measure) in the sense of Coifman and Weiss (see [C–W]). Denote S a function space of bounded measurable functions on X and S_p $(1 \le p \le \infty)$ the closure of S in $L^p(\mu)$. Assume the orthogonal projection on S_2 is given by a Calderon–Zygmund type kernel, i.e.

$$Pf(x) = \int f(y)K(x,y)\mu(dy) \tag{2.1}$$

where K satisfies the well-known condition

$$\int_{X\setminus B(y_0,2r)} |K(x,y) - K(x,y_0)|\mu(dx) < c \tag{2.2}$$

wherever

$$y_0 \in X, \quad r > 0, \quad y \in B(y_0, r) = \{y \in X \mid d(y_0, y) < r\}. \tag{2.3}$$

LEMMA 2.4: *For $t > 0, 1 < p < \infty$, the couple (S_1, S_p) is K-closed in $(L^1(\mu), L^p(\mu))$.*

Proof: Let $s \in S$ have the decomposition $s = f + g$ where $\|f\|_1 + t\|g\|_p \leq 1\,(t < 1, f \in L^1, g \in L^p)$. Put $\lambda = t^{-p'}$ and make a Calderon–Zygmund decomposition of f as $f = h + k$ where

$$|h| \leq \lambda; \tag{2.5}$$

$k = \Sigma_{j\geq 1} k_j$ where to each j a ball B_j may be associated such that

$$\Sigma\mu(B_j) < c\frac{\|f\|_1}{\lambda} = c\lambda^{-1}, \tag{2.6}$$

$$\operatorname{supp} k_j \subset B_j, \tag{2.7}$$

$$\int k_j = 0, \tag{2.8}$$

$$\Sigma\|k_j\|_1 < c \tag{2.9}$$

(c refers to a constant depending on X). The construction is well-known (based on the Vitali-covering property).

Write

$$s = (g + h) + k = P(g + h) + Pk. \tag{2.10}$$

Since P is given by a Calderon–Zygmund kernel, P is $L^P - L^p$ bounded for $1 < p < \infty$ and thus

$$\|P(g+h)\|_p \leq \|g\|_p + \|h\|_p \leq t^{-1} + \|h\|_1^{1/p}\|h\|_\infty^{1/p'} < t^{-1} + c\lambda^{1/p'} = ct^{-1} \tag{2.11}$$

using (2.5), the estimate $\|h\|_1 \leq \|f\|_1 + \|k\|_1 < c$ and the choice of λ. It remains to estimate $\|Pk\|_1$. Define $\Omega = \bigcup B_j^*$ where B_j^* refers to the doubling of B_j (= same center y_i, double radius). Write

$$\|Pk\|_1 \leq \|P(k)\chi_\Omega\|_1 + \|P(k) \cdot \chi_{X\setminus\Omega}\|_1. \tag{2.12}$$

Estimate the second term of (2.12) as

$$\begin{aligned}\sum_j \|P(k_j)X_{X\setminus B_j^*}\|_1 &= \sum_j \int_{X\setminus B_j^*} \Big| \int k_j(y)[K(x,y) - K(x,y_j)]dy\Big| dx \\ &\leq \sum_j \int |k_j(y)| \left\{ \int_{X\setminus B_j^*} |K(x,y) - K(x,y_j)|dx \right\} dy \\ &\leq c \sum_j \|k_j\|_1 \\ &< c\end{aligned} \tag{2.13}$$

using successively (2.8), (2.2), (2.9).

The only point which is not entirely straightforward is the bound on the first term of (2.12). Here we use (2.10) and write, using Hölder's inequality,

$$P(k) = k + (g+h) - P(g+h), \tag{2.14}$$

$$\|P(k)\chi_\Omega\|_1 \leq \|k\|_1 + \mu(\Omega)^{1/p'}[\|g+h\|_p + \|P(g+h)\|_p]. \tag{2.15}$$

Next, apply (2.9), (2.11) to get from (2.15)

$$\|P(k)\chi_\Omega\|_1 \leq C + Ct^{-1}\mu(\Omega)^{1/p'}. \tag{2.16}$$

Here $\mu(\Omega) \leq \Sigma\mu(B_j^*) \leq C\Sigma\mu(B_j) < C\lambda^{-1}$, by (2.6), so that (2.16) also is bounded.

Hence (2.10) yields an appropriate decomposition of s in $S_p + S_1$ in the sense that $K_t(s, S_1, S_p) \leq C$, for some constant C. This proves the lemma. ∎

Next we recall T. Wolff's interpolation theorem [W] in the real case.

THEOREM 2.17 (see [W], Th. 1): *Let A_1, A_2, A_3, A_4 be quasi-Banach spaces satisfying $A_1 \cap A_4 \subset A_2 \cap A_3$ and suppose $[A_1, A_3]_{\theta,q} = A_2$, $[A_2, A_4]_{\phi,r} = A_3$ where $0 < \theta$, $\phi < 1$, $0 < q, r \leq \infty$. Then*

$$A_2 = [A_1, A_4]_{\psi,q} \qquad \text{and} \qquad A_3 = [A_1, A_4]_{\xi,r}$$

where

$$\psi = \frac{\theta\phi}{1-\theta+\theta\phi}, \qquad \xi = \frac{\phi}{1-\theta+\theta\phi}.$$

Coming back to the proof of Theorem 1 and applying (2.17) in the $H^p(B)$ scale, it clearly suffices to consider pairs

$$(2.18) \qquad (1, p_1) \qquad p_1 < \infty,$$

$$(2.19) \qquad (p_0, \infty) \qquad p_0 > 1.$$

The Cauchy projection for the ball B_d is given by the kernel $K(x,y) = (1 - \langle x,y\rangle)^{-d}, x \in B_d, y \in \partial B_d \equiv \Sigma_{2d-1}$. Endowing Σ_{2d-1} with the non-isotropic distance $d(x,y) = |1 - \langle x,y\rangle|^{1/2}$, a homogeneous space is obtained and $K(x,y)$ is a (generalized) Calderon–Zygmund type operator, satisfying in particular (2.2). These facts are classical and we omit references. Applying Lemma 2.4 a first time, it follows that

$$(H^1(B), H^{p_1}(B)) \text{ is } K\text{-closed in } (L^1(\Sigma), L^{p_1}(\Sigma))$$

for $p_1 < \infty$. The application is immediate here. From (1.3), one concludes (1.17), (1.18) in case (2.18).

The case (2.19) is treated by Pisier's Lemma. Thus the statement

$$(2.20) \qquad (H^{p_0}(B), H^\infty(B)) \text{ is } K\text{-closed in } (L^{p_0}(\Sigma), L^\infty(\Sigma))$$

gets reduced to

$$(2.21) \qquad (H^{p_0}(B)^\perp, H^\infty(B)^\perp) \text{ is } K\text{-closed in } (L^{p_0'}(\Sigma), L^1(\Sigma)).$$

Here $H^p(B)^\perp$ stand for the subspace of elements $f \in L^{p'}(\Sigma)$ such that $\langle f,g\rangle = \int_\Sigma f\cdot\bar{g}\,d\sigma = 0$ whenever $g \in H^p$. The orthogonal projection on $H^2(B)^\perp$ is given by Id-K, satisfying (2.2) equally. Observe that for $p > 1$ the space $H^p(B)^\perp$ appears as closure in $L^{p'}(\Sigma)$ of $C^\infty(\Sigma) \cap H(B)^\perp$ using, for instance, Poisson convolution. This provides an appropriate dense subspace S, such that $H^\infty(B)^\perp = S_1$ and $H^{p_0}(B)^\perp = S_{p_0'}$ $(1 < p_0 < \infty)$. (2.21) then follows from Lemma 2.4 again.

Remarks:

(i) It is clear that the previous considerations only rely on the geometry of the boundary of the domain and consequently the unit ball $B_d \subset C^d$ may be replaced by a strictly pseudo-convex domain.

(ii) The question whether $(H^1(B_d),\ H^\infty(B_d))$ is K-closed in

$$(L^1(\Sigma_{2d-1}), L^\infty(\Sigma_{2d-1})) \quad \text{for } d \geq 2$$

is left open from the previous argument.

(iii) The arguments used in [J] and [P1] to characterize $H^p(D)$ as the **complex** interpolation space $[H_1, H^\infty]_\theta$ are different, but both of them involve vector-valued Hardy spaces. At present, it is not clear to the author how to deal with the $H^p(B_d)$-spaces, $d \geq 2$, when $p = \infty$ appears as endpoint. As will be observed later, vector-valued H^∞-spaces on the ball may behave differently than in the disc case and, particularly, the method of [J] is not applicable.

(iv) There are a variety of other models where the method of proving Theorem 1 applies equally well. For instance, one gets a simple proof of following real interpolation result for the Sobolov spaces, due to De Vore and Scherer [DV–S] (the corresponding result for complex interpolation is open, cf. [J2]).

THEOREM 3: *Denote for $d \geq 1$ and $1 \leq p \leq \infty$ by $W^{1,p}(\mathbb{T}^d)$ the closure of the trigonometric polynomials on $\mathbb{T}^d$ under the norm*

$$\|\nabla f\|_p \equiv \left(\sum_{i=1}^{d} \|\partial_i f\|_p^p\right)^{1/p} \tag{2.22}$$

where ∂_i stands for the ith partial derivative (assume $\int f = 0$ for simplicity). Then

$$W^{1,p}(\mathbb{T}^d) = [W^{1,1}(\mathbb{T}^d), W^{1,\infty}(\mathbb{T}^d)]_{\theta,p} \tag{2.23}$$

where $1/p = 1 - \theta$.

Identify the space $W^{1,p}(\pi^d)$ with the subspace S_p of the ℓ^p-sum of d copies of $L^p(\mathbb{T}^d)$, through the norm definition (2.22). Thus

$$S_p = \{\, (\partial_1 f, \ldots, \partial_d f) \mid f \in W^{1,p} \}.$$

For $p = 2$, the orthogonal projection is given by

$$P(f_1,\dots,f_d) = \left(\sum_{j=1}^{d}\left(\frac{\xi_1\xi_j}{|\xi|^2}\hat{f}_j\right)^{\vee},\dots,\sum_{j=1}^{d}\left(\frac{\xi_d\xi_j}{|\xi|^2}\hat{f}_j\right)^{\vee}\right) \tag{2.24}$$

where $\xi = (\xi_1,\dots,\xi_d) \in \mathbb{Z}^d|\{0\}$ and $^\wedge$ (resp. $^\vee$) stand for Fourier (resp. inverse Fourier) transform. The Fourier multipliers appearing in (2.24) are well-known to define convolution operators of Calderon–Zygmund type. One then follows the same scheme as for Theorem 1. A similar argument may also be given for higher order derivatives. The [DV–S] method has the advantage however of being constructive.

(v) The interpolation properties of the vector-valued scale $H^\infty_{\ell^p}(D)$ $(1 \le p \le \infty)$ fail if one replaces D by the ball $B_d, d \ge 2$ (on the polydisc). The key ingredient appears in [B2], [B3], namely the following lemma.

LEMMA 2.29 (see [B3]): *For $0 < \epsilon < 1$, $n = 1,2,\dots$, there are 1-bounded homogeneous polynomials $\{p_j\}_{j=1}^n$ on B, $d(p_j) \equiv$ degree $(p_j) = N_j$, for which the sets $\{\zeta \in \Sigma|\ |p_j(\zeta)| > \epsilon\}$ are disjoint but $\|\sum_{j=1}^n |q_j|\ \|_\infty > cn\epsilon$ whenever $\{q_j\}_{j=1}^n$ is a sequence in $H^\infty(B)$ satisfying $\|p_j - q_j\|_\infty < c$ $(1 \le j \le n)$. Here $c > 0$ is numerical (or depending on dimension d) and $\{N_j\}$ is any rapidly enough increasing sequence.*

From this, it is easily seen that $H^\infty_{\ell^r_n}$ $(2 < r < \infty)$, for instance, does not interpolate between $H^\infty_{\ell^2_n}$ and $H^\infty_{\ell^\infty_n}$. Take $\epsilon = n^{-1/r}$ in the lemma, so that

$$\|\{p_j\}_{j=1}^n\|_{H^\infty_{\ell^r_n}} \le 1. \tag{2.25}$$

If $\{q_j\}_{j=1}^n$ is an approximating sequence in $H^\infty_{\ell^\infty_n}$, in the sense that

$$\|p_j - q_j\|_\infty < c \qquad (1 \le j \le n), \tag{2.26}$$

the previous lemma asserts that

$$\|\{p_j - q_j\}_{j=1}^n\|_{H^\infty_{\ell^2_n}} \ge n^{-1/2}\|\{p_j - q_j\}_{j=1}^n\|_{H^\infty_{\ell^1_n}} > cn^{1/2-1/r} \tag{2.27}$$

is unbounded when $n \to \infty$.

Interpolation properties of vector-valued nature are closely related to the behavior of absolutely summing operators on $H^\infty(D)$, cf. [B1], [B2], [B3], [P2]. In

[B4], an example is given of a bounded linear operator $u : H^\infty(B_2) \to \ell^p$ $(2 < p < \infty)$ which has no extension to $L^\infty(\Sigma)$ and in fact is not absolutely summing. (For $\phi = 2$, the problem remains open.) The example also shows the failure of (1.19) if the disc is replaced by the complex ball.

3. Proof of Theorem 2

The general method is that used in [P2]. Denote $\mathcal{H}^\infty$ the annilator of $H^1(B)$. We prove the following

PROPOSITION 3.1: *If Y is a Banach space, $2 < q < \infty$, and $u : \mathcal{H}^\infty \to Y$ a 2-summing operator, then the following inequality holds for the q and 2-summing norms of u :*

$$\pi_q(u) \le C\pi_2(u)^\theta \cdot \|u\|^{1-\theta} \tag{3.2}$$

where $1/q = \theta/2 + (1-\theta)/\infty$ and C a constant.

It follows then from Proposition 3.1 that any linear operator from $\mathcal{H}^\infty$ into a Banach space of cotype 2 (and with bounded approximation properly, say) is 2-summing. In particular, the spaces $L^1/H^1(B), (\mathcal{H}^\infty)^*$ satisfy Grothendieck's Theorem, i.e.

$$\mathcal{L}(L^1/H^1(B), \ell^2) = \Pi_1(L^1/H^1(B), \ell^2) \tag{3.3}$$

and are of cotype 2.

As in [P2], (3.2) is deduced from the interpolation inequality

$$\|f\|_{L^1/H^1(\ell^p)} \le C\|f\|^\theta_{L^1/H^1(\ell^2)}\|f\|^{1-\theta}_{L^1/H^1(\ell^1)} \tag{3.4}$$

where $1 < p < 2$, $1/p = \theta/2 + (1-\theta)/1$, $f \in L^1/H^1(\ell^p)$.

This last inequality follows (using the duality lemma [P2, Prop. 1]) from

PROPOSITION 3.5: *There is a constant C such that for all $t > 0$ and $f \in H^1_{\ell^1}(B) + H^1_{\ell^2}(B)$, we have*

$$K_t\left(f, H^1_{\ell^1}(B), H^1_{\ell^2}(B)\right) \le CK_t\left(f, L^1_{\ell^1}(S), L^1_{\ell^2}(S)\right). \tag{3.6}$$

The rest of this section is devoted to the proof of (3.6). We will use the 1-variable result, i.e. Prof. 2 of [P2], together with real-variable techniques.

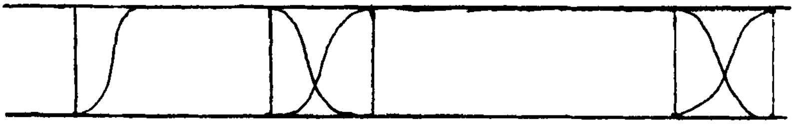
Parametrizing $\zeta \in S = \partial B_2$ as $\zeta = (\sqrt{\rho}e^{2\pi i\theta}, \sqrt{1-\rho}e^{2\pi i\psi})$, the surface element $d\sigma$ is given by $d\sigma = d\rho d\theta d\psi$. If $f \in H^2(B)$, there is the representation

$$f = \sum_{k\geq 0} P_k f \tag{3.7}$$

where P_k is the orthogonal projection on the $(k+1)$-dimensional space of homogeneous polynomials of degree k, i.e. $W_k = [z^j w^{k-j} | 0 \leq j \leq k]$. Letting P_k act on $L^2(S)$, one has

$$P_k f(\zeta) = \frac{\int f(\eta)\langle\zeta,\eta\rangle^k \sigma(d\eta)}{\int |\langle\underline{1},\eta\rangle|^{2k}\sigma(d\eta)} = (k+1)\int f(\eta)\langle\zeta,\eta\rangle^k\sigma(d\eta). \tag{3.8}$$

We introduce a system of (smooth) diadic multipliers, similarly as in the disc case



Thus the rth multiplier $C^{(r)}$ is essentially supported by the rth diadic interval and they form a partition of unity of $\mathbb{Z}$. Define the operators

$$Q = Q^{(r)} = \sum C_k^{(r)} P_k. \tag{3.9}$$

We investigate first the properties of the corresponding kernel-function, which we also denote $Q(\zeta,\eta)$. Thus from (3.8)

$$Q(\zeta,\eta) = \sum (k+1) C_k \langle\zeta,\eta\rangle^k, \tag{3.10}$$

and using the above parameterization,

$$Q(\zeta,1) = \sum (k+1) C_k \rho^{k/2} e^{2\pi i k\theta}. \tag{3.11}$$

Assuming the multiplier $C^{(r)}$ sufficiently smooth, one clearly has the following pointwise estimates (denoting $N = 2^r$) :

$$|Q(\zeta,\underline{1})| \lesssim N^2(1-\rho)^{N/4}, \tag{3.12}$$

$$|Q(\zeta,\underline{1})| \lesssim |\theta|^{-10}\|\partial_t^{(10)}\{(t+1)C_t\rho^{t/2}\}\|_{L^1([N/2,2N],dt)}$$

$$\lesssim |\theta|^{-10}\sup_{\substack{i+j=8\\ 0\le\rho\le 1}}\left(\frac{1}{N}\right)^i(1-\rho)^j\rho^{n/4} \lesssim N^{-8}|\theta|^{-10}. \tag{3.13}$$

Denote as usual $d(a,b)=|1-\langle a,b\rangle|^{1/2}$ the non-isotropic distance. Thus

$$d(\zeta,\underline{1}) = |1-\zeta_1|^{1/2} = |1-\sqrt{\rho}e^{2\pi i\theta}|^{1/2} \sim (1-\rho)^{1/2}+|\theta|^{1/2}. \tag{3.14}$$

It follows then from (3.12), (3.13) that if $d(\zeta,\underline{1}) > A/\sqrt{N}$, $A\ge 1$, then $|Q(\zeta,\underline{1})| \lesssim A^{-18}N^2$. Hence, by invariance under unitary group

$$d(\zeta,\eta) > A/\sqrt{N} \Rightarrow |Q(\zeta,\eta)| \lesssim N^2\cdot A^{-18}. \tag{3.15}$$

In follows in particular that Q extends to a bounded operator on all $L^p(S)$-spaces $(1\le p\le\infty)$, but the more precise decay estimate (3.15) will also be of importance later on.

Recall also the Poisson kernel $P(z,\zeta)$, $z\in B_2$, $\zeta\in S$,

$$P(z,\zeta) = \frac{(1-|z|^2)^2}{|1-\langle z,\zeta\rangle|^4}. \tag{3.16}$$

Thus if $1-|z|\sim 1/N$ and $d(\zeta,z) > A/\sqrt{N}$, there is the estimate

$$P(z,\zeta) \lesssim N^{-2}\left(\frac{A^8}{N^4}\right)^{-1} = N^2\cdot A^{-8} \tag{3.17}$$

Returning to the problem of proving (3.6), assume

$$f(\zeta) = F^{(1)}(\zeta)+F^{(2)}(\zeta), \qquad \zeta\in S \tag{3.18}$$

where $F^{(i)}\in L^1_{\ell^i}(S)$ and

$$\|F^{(1)}\|_{L^1_{\ell^1}} \qquad \le 1, \tag{3.19}$$

$$\|F^{(2)}\|_{L^1_{\ell^2}} \qquad \le t^{-1}. \tag{3.20}$$

Taking $z\in\partial D$ one has, following decomposition of the slice function f_ζ,

$$f_\zeta(z) = F^{(1)}(z\xi)+F^{(2)}(z\xi). \tag{3.21}$$

At this point, use Prop. 2 of [P2], thus the fact that $(H^1_{\ell^1}(D), H^1_{\ell^2}(D))$ is K-closed in $(L^1_{\ell^1}(S), (L^1_{\ell^2}(S))$. This yields a new decomposition

$$f_\zeta(z) = G^{(1)}(\zeta, z) + G^{(2)}(\zeta, z) \tag{3.22}$$

preserving analyticity in $z \in D$ and where, for $\zeta \in S$,

$$\|G^{(1)}(\zeta, \cdot)\|_{H^1_{\ell^1}(D)} \le C \int_0^1 \|F^{(1)}(e^{2\pi i\theta}\zeta)\|_{\ell^1} d\theta, \tag{3.23}$$

$$\|G^{(2)}(\zeta, \cdot)\|_{H^1_{\ell^2}(D)} \le C \int_0^1 \|F^{(2)}(e^{2\pi i\theta}\zeta)\|_{\ell^2} d\theta. \tag{3.24}$$

Integrating (3.23), (3.24) over $\zeta \in S$, one gets from (3.15), (3.20)

$$\|G^{(1)}\|_{L^1_{H^1_{\ell^1}(D)}(S)} < C, \tag{3.25}$$

$$\|G^{(2)}\|_{L^1_{H^1_{\ell^2}(D)}(S)} < Ct^{-1}. \tag{3.26}$$

Exploiting z-analyticity and the Stein tent-space multipliers acting on $H^1(D)$, one finds a splitting (using previous notations)

$$Q^{(r)} f(z\zeta) = G^{(1)}_r(\zeta, z) + G^{(2)}_r(\zeta, z), \qquad r = 0, 1, 2, \dots \tag{3.27}$$

where $\{G^{(1)}_r\}, \{G^{(2)}_r\}$ satisfy the square function estimates

$$\|(\sum_r |G^{(1)}_r|^2)^{1/2}\|_{L^1_{\ell^1}(S\times\mathbb{T})} \le c, \tag{3.28}$$

$$\|(\sum_r |G^{(2)}_r|^2)^{1/2}\|_{L^1_{\ell^2}(S\times\mathbb{T})} \le ct^{-1}. \tag{3.29}$$

Next, eliminate the z-variable, defining

$$g^i_r(\zeta) = \int_0^1 G^i_r(\zeta e^{-2\pi i\theta}, e^{2\pi i\theta}) d\theta \qquad (i = 1, 2). \tag{3.30}$$

From (3.27), (3.28), (3.29), one gets

$$f = \sum_{r=0}^{\infty} f_r; \qquad f_r \equiv Q^{(r)} f = g^{(1)}_r + g^{(2)}_r, \qquad r = 0, 1, \dots, \tag{3.31}$$

$$\|(\sum |g_r^{(1)}|^2)^{1/2}\|_{L^1_{\ell 1}(S)} < c, \tag{3.32}$$

$$\|(\sum |g_r^{(2)}|^2)^{1/2}\|_{L^1_{\ell 2}(S)} < ct^{-1}. \tag{3.33}$$

It is, of course, our aim to replace the $g_r^{(i)}$ by analytic functions.

Assume $Q^{(r)}$ redefined such that

$$Q^{(r)} f_r = f_r \tag{3.34}$$

(by taking a bit larger bases).

Next define for each r an operator $K = K^{(r)}$ essentially appearing as an expectation operator for a partition of S in sets of non-isotropic size $\sim 2^{-r/2}$. More precisely

$$Kf = \sum_i \langle f, P_i \rangle \chi_i \tag{3.35}$$

where

(3.36) χ_i is indicator function of a set $\cup_i \subset S$, the $\{\cup_i\}$ are disjoint,

$$\operatorname{diam} \cup_i < \delta \cdot 2^{-r/2} \qquad \text{and} \qquad \sigma(\cup_i) \sim 4^{-r};$$

(3.37) $P_i(\zeta) = P(z_i, \zeta)$ where $1 - |z_i| = \delta \cdot 2^{-r}$ and $\cup_i \subset B(z_i, \delta \cdot 2^{-r/2})$.

The number $0 < \delta < 1$ will be some sufficiently small constant. More specifically, δ is chosen such that (we leave details to the reader)

$$\|(I - K^{(r)})Q^{(r)}\|_{r \to \infty} < \epsilon \tag{3.38}$$

where $\epsilon > 0$ will be specified later.

Fix r. One has from (3.34)

$$f_r = Q[I - (I-K)Q]^{-1} K f_r. \tag{3.39}$$

Hence, by definition of K and using a Neumann-series expansion for the inverse,

$$f_r = \sum_{j=0}^{\infty} \left(\sum_i \langle f_r, P_i \rangle Q[(I-K)Q]^j (\chi_i) \right) = \sum_{j=0}^{\infty} \left(\sum_i \langle f_r, P_i \rangle \chi_{ij} \right) \tag{3.40}$$

where

$$\chi_{ij} = \chi_{ij}^{(r)} = Q[(I-K)Q]^j \chi_i \tag{3.41}$$

is an analytic polynomial in the space $[W_k; k \sim 2^r]$.

We establish some properties of χ_{ij} for later use. First, from (3.38)

$$|\chi_{ij}| < \epsilon^j. \tag{3.42}$$

Observe that as a consequence of (3.17), there is the decay estimate

$$K(\zeta,\eta) < c_\delta N^2 A^{-8} \qquad \text{if } d(\zeta,\eta) > AN^{-1/2} \qquad (N = 2^r) \tag{3.43}$$

on the kernel

$$K(\zeta,\eta) = \sum_i P(z_i,\eta)\chi_i(\zeta). \tag{3.44}$$

This operator has $L^1 - L^1$ and $L^\infty - L^\infty$ bounds by some constant (**independent of** δ). Write

$$\begin{aligned} &Q[(I-K)Q]^j(\zeta,\eta) = \\ &\int_{S\times\cdots\times S} Q(\zeta,\xi_1)(I-K)(\xi_1,\xi_2)Q(\xi_2,\xi_3)(I-K)(\xi_3,\xi_4)\cdots Q(\xi_{2j},\eta) \end{aligned} \tag{3.45}$$

and use (3.15), (3.43) to get the decay estimate

$$|Q[(I-K)Q]^j(\zeta,\eta)| < C^j C_\delta A^{-8} N^2 \qquad \text{if} \quad d(\zeta,\eta) > A/\sqrt{N}, \quad A > 1. \tag{3.46}$$

Here C is a numerical constant.

Consequently, by (3.41)

$$|\chi_{ij}(\zeta)| < C^j C_\delta A^{-8} \qquad \text{provided} \quad d(\zeta, z_i) > A/\sqrt{N} \tag{3.47}$$

and using (3.42), writing $|\chi_{ij}| = |\chi_{ij}|^{1/8}|\chi_{ij}|^{7/8}$, a suitable choice of ϵ gives

$$|\chi_{ij}(\zeta) \lesssim 10^{-j} A^{-7} \qquad \text{provided} \quad d(\zeta, z_i) > A/\sqrt{N}. \tag{3.48}$$

We come back to (3.40). Choose some number γ,

$$0 < \gamma < 1, \tag{3.49}$$

sufficiently close to 1 (to be specified). Recall (3.31),

$$f_r = g_r^{(1)} + g_r^{(2)}, \tag{3.50}$$

which are vector-valued functions (ranging in sequence space), which s-coordinate is denoted $f_{r,s}, g_{r,s}^{(i)}$ ($i = 1,2$). Since $f_{r,s}$ is holomorphic on B, it follows from subharmonicity that

$$\langle |f_{r,s}|^\gamma, P_i\rangle \geq |f_{r,s}(z_i)|^\gamma \tag{3.51}$$

and hence, by (3.50),

$$|\langle f_{r,s} P_i\rangle| < 2\left(\langle |g_{r,s}^{(1)}|^\gamma, P_i\rangle\right)^{1/\gamma} + 2\left(\langle |g_{r,s}^{(2)}|^\gamma, P_i\rangle\right)^{1/\gamma}. \tag{3.52}$$

Therefore, one may find complex numbers $a_{r,s,i}^{(1)}, a_{r,s,i}^{(2)}$ bounded by 2 such that

$$\langle f_{r,s}, P_i\rangle = a_{r,s,i}^{(1)}\left(\langle |g_{r,s}^{(1)}|^\gamma, P_i\rangle\right)^{1/\gamma} + a_{r,s,i}^{(2)}\left(\langle |g_{r,s}^{(2)}|^\gamma, P_i\rangle\right)^{1/\gamma}. \tag{3.53}$$

Substitute (3.53) in (3.40) to get the decomposition

$$\sum_{j=0}^{\infty}\sum_i a_{r,s,i}^{(1)}\left(\langle |g_{r,s}^{(1)}|^\gamma, P_i\rangle\right)^{1/\gamma}\chi_{ij} + \sum_{j=0}^{\infty}\sum_i a_{r,s,i}^{(2)}\left(\langle |g_{r,s}^{(2)}|^\gamma, P_i\rangle\right)^{1/\gamma}\chi_{ij}. \tag{3.54}$$

Here both systems $\{P_i\}, \{\chi_{ij}\}$ depend on r. Summing over r yields the following analytic decomposition of $\{f_s\} \equiv f = f^{(1)} + f^{(2)}$ where

$$f_s^{(1)} = \sum_r \left\{\sum_j \sum_i a_{r,s,i}^{(1)} \langle |g_{r,s}^{(1)}|^\gamma, P_i^{(r)}\rangle^{1/\gamma}\chi_{ij}^{(r)}\right\}, \tag{3.55}$$

$$f_s^{(2)} = \sum_r \left\{\sum_j \sum_i a_{r,s,i}^{(2)} \langle |g_{r,s}^{(2)}|^\gamma, P_i^{(r)}\rangle^{1/\gamma}\chi_{ij}^{(r)}\right\}. \tag{3.56}$$

We estimate $\|f^{(1)}\|_{H^1_{\ell^1}}$ and $\|f^{(2)}\|_{H^1_{\ell^2}}$. Because $\chi_{ij}^{(r)} \in \{W_k; k \sim 2^r]$, one may introduce square functions and

$$\begin{aligned}
\|f^{(1)}\|_{L^1_{\ell^1}} &\sim \left\| \left(\sum_r \Big|\sum_j \sum_i a_{r,s,i}^{(1)} \langle |g_{r,s}^{(1)}|^\gamma, P_i^{(r)}\rangle^{1/\gamma}\chi_{ij}^{(r)}\Big|^2\right)^{1/2} \right\|_{L^1_{\ell^1}} \\
&< \sum_j \left\| \left(\sum_r \left(\sum_i \langle |g_{r,s}^{(1)}|^\gamma, P_i^{(r)}\rangle |\chi_{ij}^{(r)}|^\gamma\right)^{2/\gamma}\right)^{1/2} \right\|_{L^1_{\ell_1}} \\
&= \sum_j \left\| \left(\sum_r \left(\sum_i \langle |g_{r,s}^{(1)}|^\gamma, P_i^{(r)}\rangle |\chi_{ij}^{(r)}|^\gamma\right)^{2/\gamma}\right)^{\gamma/2} \right\|^{1/\gamma}_{L^{1/\gamma}_{\ell^{1/\gamma}}}.
\end{aligned} \tag{3.57}$$

Denote $p = 1/\gamma > 1, q = 2/\gamma$ and $u_{r,s} = |g^{(1)}_{r,s}|^\gamma$.

Consider the individual sums

$$\sum_i \langle u_{r,s}, P_i^{(r)}\rangle |\chi_{ij}^{(r)}|^\gamma \tag{3.58}$$

on which we make a pointwise estimate.

From (3.48), there is the decay estimate

$$|\chi_{ij}^{(r)}|^\gamma(\zeta) < 5^{-j}\cdot A^{-6} \qquad \text{provided} \quad d(\zeta, z_i) > A/\sqrt{N}, \quad A > 1 \tag{3.59}$$

(choosing γ close enough to 1).

Fix a point $\zeta \in S$, $d(\zeta, z_{i_0}) \sim 1/\sqrt{N}$ for some i_0. If $A > 1$ ranges over dyadique numbers, the contribution

$$\sum_{d(z_i, z_{i_0})\sim A/\sqrt{N}} \langle u_{r,s}, P_i^{(r)}\rangle |\chi_{ij}^{(r)}|^\gamma(\zeta) \tag{3.60}$$

is easily seen (using (3.59)) to be bounded by

$$5^{-j}A^{-6} \sum_{d(z_i, z_{i_0})\sim A/\sqrt{N}} \langle u_{r,s}, P_i^{(r)}\rangle < 5^{-j}A^{-6}\frac{\sigma\left(B(\underline{1}, A/\sqrt{N})\right)}{\sigma\left(B(\underline{1}, 1/\sqrt{N})\right)} u^*_{r,s}(\zeta) \tag{3.61}$$

where $u^*(\zeta)$ stands for the Hardy–Littlewood maximal function of u (with respect to the non-isotropic structure). Hence (3.60) is bounded by

$$5^{-j}A^{-2}u^*_{r,s}(\zeta) \tag{3.62}$$

and (3.58) is bounded by $C\cdot 5^{-j}u^*_{r,s}(\zeta)$, summing (3.62) over A. Substitution of this in (3.57) gives the bound

$$\sum_j 5^{-j}\left\| \left(\sum_r (u^*_{r,s})^q\right)^{1/q} \right\|^p_{L^p_{\ell^p}}. \tag{3.63}$$

Since there are no $1, \infty$ endpoints, (3.63) is further bounded by

$$C\left\| \left(\sum_r (u_{r,s})^q\right)^{1/q} \right\|^p_{L^p_{\ell^p}} = C\left\| \left(\sum_r (g^{(1)}_{r,s})^2\right)^{1/2} \right\|_{L^1_{\ell^1}} < C \tag{3.64}$$

by definition of $u_{r,s}$ and inequality (3.32). The passage from (3.63) to the left member of (3.64) results from a classical vector-valued maximal inequality for mixed-norm L^p-spaces (or, more generally, UMD-lattices, cf. [B5]).

The estimate on $\|f^{(2)}\|_{H^1_{\ell^2}}$ is completely similar and (3.33) implies $\|f^{(2)}\|_{H^1_{\ell^2}} < ct^{-1}$, completing the proof of Proposition 3.5. ∎

Appendix: The Quotient Spaces $L^1(\mathbb{T})/H^1(\mathbb{D})$ and $L^1(S)/H^1(B_m)$ are Non-isomorphic Banach Spaces ($m \geq 2$)

This question appears rather naturally from the proof of Theorem 2 in the paper and there is also the known fact that $H^1(D)$ and $H^1(B_m)$ are isomorphic spaces (which is not true in the polydisc case). This $H^1(D) - H^1(B_m)$ isomorphism may not be extended, however, to the corresponding L^1-spaces.

The proof of the claim made in the title of the appendix is of infinite-dimensional nature. Here are the ingredients:

(i) Let $s : L^1(\mathbb{T})/H^1 \to L^1(S)/H^1$, $t : L^1(S)/H^1 \to L^1(\mathbb{T})/H^1$ be inverse isomorphisms. Then there are liftings $\tilde{s}, \tilde{t}$ making the following diagram commute:

$$\begin{array}{ccccc} L^1(\mathbb{T}) & \xrightarrow{\tilde{s}} & L^1(S) & \xrightarrow{\tilde{t}} & L^1(\mathbb{T}) \\ q\downarrow & & q'\downarrow & & q\downarrow \\ L^1(\mathbb{T})/H^1 & \xrightarrow{s} & L^1(S)/H^1 & \xrightarrow{t} & L^1(\mathbb{T})/H^1 \end{array}$$

where q, q' are the quotient maps. The only point to observe here is that in constructing the liftings using at the end compactness (naturally giving rise to operators ranging in the space of measures), one ends up with L^1-functions. The reason for this is that $H^1(D)$ (resp. $H^1(B)$) is ω^* -closed in $M(\mathbb{T})$ (resp. $M(S)$).

(ii) Let $T : L^1(\mu) \to L^1(\nu)$ be a bounded operator and T^* its adjoint. Then there is a measure $\nu_1 \ll \nu$ such that T^* extends from $L^1(\nu_1)$ to $L^1(\mu)$. In fact, we may take

$$\frac{d\nu_1}{d\nu} = \sup_{\varphi \in L^\infty(\mu), |\varphi| \leq 1} |T\varphi|.$$

The argument is routine.

(iii) Apply (ii) repetitively to the operators $\tilde{s}, \tilde{t}$ from (i). Thus $\mu_0 = \sigma$ (invariant measure on $\mathbb{T}$), $\nu_0 = \sigma'$ (invariant measure on S) $\Rightarrow \nu_1$ on S such that $(\tilde{s})^*$ is $\nu_1 - \mu_0$ bounded (may take $\nu_1 \ll \sigma', d\nu_1/d\sigma' > 1$). Since $\tilde{t}$ is $\nu_1 - \sigma$ bounded $\Rightarrow \mu_1$ on $\mathbb{T}$ such that $(\tilde{t})^*$ is $\mu_1 - \nu_1$ bounded ($\mu_1 \ll \sigma, d\mu_1/d\sigma > 1$). Then $\tilde{s}$ is $\mu_1 - \sigma'$ bounded $\Rightarrow \nu_2$ on S, etc.

At each state

$(\tilde{s})^*$ is $\nu_{j+1} - \mu_j$ bounded,

$(\tilde{t})^*$ is $\mu_j - \nu_j$ bounded.

Obviously, the norms of these measures grow at most exponentially. Hence, defining for a suitable constant C (depending on $\|\tilde{s}\|$, $\|\tilde{t}\|$)

$$\mu = \sum_{j\geq 0} C^{-j}\mu_j, \qquad \nu = \sum_{j \geq 0} C^{-j}\nu_j$$

on $\mathbb{T}, S$ respectively,

$(\tilde{s})^*$ gets a bounded extension $\bar{s}: L^1(S,v) \to L^1(\mathbb{T},\mu)$,

$(\tilde{t})^*$ gets a bounded extension $\bar{t}: L^1(\mathbb{T},\mu) \to L^1(S,\nu)$.

Moreover $\mu \ll \sigma, \nu \ll \sigma'$ and $d\mu/d\sigma, d\nu/d\sigma' \geq 1$.

Denoting

$$A_{\mathbb{T}} = (H^1(D))^{\perp}, \qquad A_S = (H^1(B))^{\perp}$$

the annilators and

$$A^1(\mathbb{T},\mu), \qquad A^1(S,\nu)$$

their respective closures in $L^1(\mathbb{T},\nu), L^1(S,\nu)$, the following diagram is clearly obtained:

$$\begin{array}{ccccc} L^1(S,v) & \xrightarrow{\bar{s}} & L^1(\mathbb{T},\mu) & \xrightarrow{\bar{t}} & L^1(S,\nu) \\ \nearrow & & \nearrow & & \nearrow \\ A^1(S,\nu) & \xrightarrow{\bar{s}} & A^1(\mathbb{T},\mu) & \xrightarrow{\bar{t}} & A^1(S,\nu) \end{array} \qquad \bar{t} \circ \bar{s} = \mathrm{Id}$$

(iv) Clearly $H^1(D)^{\perp} = \overline{H_0^{\infty}(D)}$ and $A^1(\mathbb{T},\mu)$ is the closure in $L^1(\mathbb{T},\mu)$ of $\{e^{2\mathbb{T} in\theta}, n < 0\}$, which as a Banach space is isomorphic to the usual $H^1(\mathbb{T})$ space. This is because of the properties of μ (see [Pe]):

$$\mu \ll \sigma \quad \text{and} \quad \frac{d\mu}{d\sigma} \geq 1 \Rightarrow \int_{\mathbb{T}} \log\left|\frac{d\mu}{d\sigma}\right| > -\infty.$$

On the other hand, $A^1(S,\nu)$ contains the $L^1(S,\nu)$-closure of the linear space

$$\{|z|^j - \int_S |z|^j;\ j \geq 0\}$$

which is of codim 1 in the $L^1(\nu)$-closure of

$$(*) \qquad \{|z|^j \mid j \geq 0\}.$$

Since $(*)$ is a real algebra, the closure in $L^1(\nu)$ is an L^1-space and hence $A^1(S,\nu)$ contains L^1 as a subspace. The conclusion is that one would get a factorization of the identity on L^1 through H^1, which is of course a contradiction.

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